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A proof of the peak polynomial positivity conjecture

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Abstract. We say that a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ has a peak at index i if $\pi_{i-1} < \pi_i > \pi_{i+1}$. Let $P(\pi)$ denote the set of indices where π has a peak. Given a set S of positive integers, we define $P(S;n) = \{\pi \in \mathfrak{S}_n : P(\pi) = S\}$. In 2013 Billey, Burdzy, and Sagan showed that for subsets of positive integers S and sufficiently large n, $|P(S;n)| = p_S(n)2^{n-|S|-1}$ where $p_S(x)$ is a polynomial depending on S. They gave a recursive formula for $p_S(x)$ involving an alternating sum, and they conjectured that the coefficients of $p_S(x)$ expanded in a binomial coefficient basis centered at $\max(S)$ are all nonnegative. In this paper we introduce a new recursive formula for $|P(S;n)|$ without alternating sums and we use this recursion to prove that their conjecture is true.

Keywords: binomial coefficient, peaks, peak polynomial, permutation, positivity conjecture.

1 Introduction

Let $[n] := \{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n$ denote the symmetric group on n letters. Let $\pi = \pi_1\pi_2\cdots\pi_n$ denote the one-line notation for $\pi \in \mathfrak{S}_n$. We say that π has a *peak* at index i if $\pi_{i-1} < \pi_i > \pi_{i+1}$ and define the peak set of a permutation π to be the set:

$$P(\pi) = \{i \in [n] \mid \pi \text{ has a peak at } i\}.$$

Given a subset $S \subseteq [n]$ we denote the set of all permutations with peak set S by

$$P(S;n) = \{\pi \in \mathfrak{S}_n \mid P(\pi) = S\}.$$

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Whenever $P(S;n) \neq \emptyset$, we say $S \subseteq [n]$ is *n-admissible* or simply *admissible* when the n is understood. If S is n-admissible, then it is k-admissible for any $k \geq n$.

Billey, Burdzy, and Sagan first studied the subsets $P(S;n) \subseteq \mathfrak{S}_n$ for n-admissible sets S in 2013 [2]. Their work was motivated by a problem in probability theory which explored the relationship between mass distribution on graphs and random permutations with specific peak sets [4]. One of their foundational results established that for an n-admissible set S

$$|P(S;n)| \ = \ p_S(n)2^{n-|S|-1} \tag{1.1}$$

where $p_S(x)$ is a polynomial depending on S, which they called the *peak polynomial* of S. It was shown that $p_S(x)$ has degree $\max(S) - 1$ when $S \neq \emptyset$, $p_S(x) = 1$ when $S = \emptyset$, $p_S(x) = 0$ when S is non-admissible, and that $p_S(x)$ takes on integral values when evaluated at integers [2, Theorem 1]. Similar observations were made for peak polynomials in other classical Coxeter groups (see the work of Castro-Velez, Diaz-Lopez, Orellana, Pastrana, and Zevallos [9] and Diaz-Lopez, Harris, Insko, and Perez-Lavin [11]). Using the method of finite differences, Billey, Burdzy, and Sagan gave closed formulas for the peak polynomials $p_S(x)$ in various special cases. The *finite forward difference* operator Δ is a linear operator defined by $(\Delta f)(x) = f(x+1) - f(x)$. Iterating this operator gives higher order differences defined by

$$(\Delta^j f)(x) = (\Delta^{j-1} f)(x+1) - (\Delta^{j-1} f)(x),$$

where $(\Delta^0 f)(x) = f(x)$. Using Newton's forward difference formula, Billey, Burdzy, and Sagan expanded $p_S(x)$ in the binomial basis centered at k as

$$p_S(x) = \sum_{j=0}^{\max(S)} (\Delta^j p_S)(k) \binom{x-k}{j} \tag{1.2}$$

and conjectured that for any admissible set S with $m = \max(S)$ each coefficient $(\Delta^j p_S)(m)$ is a positive integer for $1 \leq j \leq m-1$ [2, Conjecture 14]. This conjecture has become known as the *positivity conjecture* for peak polynomials.

Example 1.1. Below is a table of forward differences for the peak polynomial $p_{\{4,6\}}(x)$. The (j,k) entry in this table is the coefficient $(\Delta^j p_S)(k)$ of $\binom{x-k}{j}$ in the expansion of $p_S(x)$ in the binomial basis centered at k.

For example, we expand $p_{\{4,6\}}(x)$ in the binomial bases centered at 0 and 6 as

$$\begin{aligned} p_{\{4,6\}}(x) &= 4\binom{x}{0} - 2\binom{x}{1} + 2\binom{x}{2} - 2\binom{x}{3} + 0\binom{x}{4} + 3\binom{x}{5} + 0\binom{x}{6} \\ &= 0\binom{x-6}{0} + 25\binom{x-6}{1} + 50\binom{x-6}{2} + 43\binom{x-6}{3} + 18\binom{x-6}{4} + 3\binom{x-6}{5} + 0\binom{x-6}{6}. \end{aligned}$$

Billey, Burdzy, and Sagan proved the positivity conjecture holds when $|S| \leq 1$ [2, Proposition 16], verified it computationally for all 2^m subsets containing a largest value

j, k	0	1	2	3	4	5	6
0	4	2	2	2	0	-3	0
1	-2	0	0	-2	-3	3	25
2	2	0	-2	-1	6	22	50
3	-2	-2	1	7	16	28	43
4	0	3	6	9	12	15	18
5	3	3	3	3	3	3	3
6	0	0	0	0	0	0	0

Table 1: Forward difference table for the peak polynomial $p_{\{4,6\}}(x)$

$m = \max(S) = 20$, and showed that $p_S(m) = 0$ for any set S [2, Lemma 15]. In 2014, Billey, Fahrbach, and Talmage posed a stronger conjecture bounding the moduli of the roots of $p_S(x)$, which they checked numerically for all peak sets with $\max(S) \leq 15$ [3, Conjecture 1.6]. They also discovered a computationally efficient recursive algorithm for computing $p_S(x)$, and showed that $p_S(k) > 0$ for $k > m$ and that the positivity conjecture holds in several special cases, including when the position of the last peak of S is three more than the position of the penultimate peak [3, Lemmas 4.4 and 3.9].

Our main result is the following theorem, which proves the positivity conjecture in all cases.

Theorem 1.2. *If $S \subseteq [n]$ is a nonempty admissible set with $m = \max(S)$, then $(\Delta^j p_S)(k) > 0$ for all $1 \leq j \leq m-1$ and $k \geq m$, and $(\Delta^m p_S)(x) = 0$.*

We prove Theorem 1.2 at the end of Section 2. As a consequence of this theorem and (1.2), if S is an n-admissible set, then the coefficients $\{(\Delta^j p_S)(k)\}_{j=1}^{m-1}$ of $p_S(x)$ when expressed in the binomial basis centered at k are positive whenever $k \geq m$. Positivity of coefficients in a given binomial basis is a phenomenon that occurs throughout combinatorics. A particular illuminating example comes from Ehrhart theory. For a d-dimensional integral convex polytope P, recall that $i_P(n)$ is the number of integer points in the n-th dilation of P. Ehrhart proved that $i_P(n)$ is a polynomial in n of degree d, so classical techniques in generating functions establish that $i_P(n) = \sum_{j=0}^{d} h_j^* \binom{n+d-j}{d}$ for complex values h_j^*, see [5]. The vector $(h_0^*, h_1^*, \ldots, h_d^*)$ is called the h^*-vector of P, and a celebrated theorem of Stanley confirms that h_j^* are nonnegative integers for all j, [14, Theorem 2.1].

In addition to positivity, we have verified that the coefficients $(\Delta^j p_S)(m)$ are log-concave in j for all admissible sets S with $m = \max(S) \leq 20$, and we suspect that log-concavity holds in general. We note that log-concavity along with our positivity result would imply the unimodality of the coefficients $(\Delta^j p_S)(m)$ for $1 \leq j \leq m-1$. If unimodality is not true in general, a related problem would be classifying peak sets for which unimodality holds. Such problems are a major theme throughout combinatorics (for instance, they are central in Ehrhart theory [5]) and could lead to many interesting and fruitful combinatorial questions.

In addition, Theorem 1.2 provides supporting evidence for Billey, Fahrbach, and Talmage's stronger conjecture bounding the moduli of the zeros of peak polynomials [3, Conjecture 1.6]. After stating that conjecture, they noted that Ehrhart, and Hilbert polynomials are all examples of polynomials with integer coefficients (in some basis) whose roots are bounded in the complex plane [1, 5, 6, 7, 8, 12, 13]. Their conjecture suggests that peak polynomials fit into the family of polynomials sharing these properties.

2 Peak polynomial positivity

We begin with a definition that is used throughout the rest of this paper.

Definition 2.1. Let $S = \{i_1, i_2, \ldots, i_s\} \subseteq [n]$ with $i_1 < i_2 < \ldots < i_s$ be an n-admissible set, and hence $P(S;n) \neq \emptyset$. For $1 \leq \ell \leq s$ define

$$\begin{aligned} S_{i_\ell} &= \{i_1, i_2, \ldots, i_{\ell-1}, i_\ell - 1, i_{\ell+1} - 1, i_{\ell+2} - 1, \ldots, i_s - 1\}, \\ \widehat{S}_{i_\ell} &= \{i_1, i_2, \ldots, i_{\ell-1}, \widehat{i_\ell}, i_{\ell+1} - 1, i_{\ell+2} - 1, \ldots, i_s - 1\}, \end{aligned}$$

where the notation $\widehat{i_\ell}$ means that the element i_ℓ has been omitted from the set.

In general, the sets S_{i_ℓ} might not be n-admissible as they may contain two adjacent integers when $i_\ell - 1 = i_{\ell-1} + 1$. However, the sets $\widehat{S}_{i_\ell}$ are always n-admissible.

Example 2.2. If $S = \{3, 5, 8\} \subseteq [9]$, then

$$\begin{aligned} &S_3 = \{2,4,7\}, \quad &&S_5 = \{3,4,7\}, \quad &&S_8 = \{3,5,7\}, \\ &\widehat{S}_3 = \{4,7\}, &&\widehat{S}_5 = \{3,7\}, &&\widehat{S}_8 = \{3,5\}. \end{aligned}$$

The sets $S_3, S_8, \widehat{S}_3, \widehat{S}_5, \widehat{S}_8$ are 9-admissible whereas S_5 is not.

Our first result describes a recursive construction of the set $P(S; q+1)$ from disjoint subsets in $\mathfrak{S}_q$.

Theorem 2.3. *Let $S = \{i_1, i_2, \ldots, i_s\} \subseteq [n]$ with $i_1 < i_2 < \ldots < i_s$ be a nonempty n-admissible set. Then for $q \geq \max(S)$*

$$|P(S; q+1)| = 2|P(S;q)| + 2\sum_{\ell=1}^{s} |P(S_{i_\ell}; q)| + \sum_{\ell=1}^{s} |P(\widehat{S}_{i_\ell}; q)|. \tag{2.1}$$

Proof. We recursively build all permutations in $P(S;q+1) \subseteq \mathfrak{S}_{q+1}$ from permutations in $\mathfrak{S}_q$ by inserting the number $q+1$ (in different positions) in the permutations of $\mathfrak{S}_q$. Let $\pi = \pi_1 \cdots \pi_q$ be a permutation in $\mathfrak{S}_q$ and consider the following five cases:

Case 1: If $\pi \in P(S;q)$, then by inserting $q+1$ after π_q we create the permutation

$$\hat{\pi} = \pi_1\pi_2 \cdots \pi_q(q+1) \in P(S;q+1).$$

Case 2: If $\pi \in P(S;q)$, then by inserting $q+1$ between π_{i_s-1} and π_{i_s} we create the permutation

$$\hat{\pi} = \pi_1 \cdots \pi_{i_s-1}(q+1)\pi_{i_s} \cdots \pi_q \in P(S;q+1).$$

Case 3: If $\pi \in P(S_{i_\ell};q)$ for any $1 \leq \ell \leq s$, then by inserting $q+1$ between $\pi_{i_\ell-1}$ and π_{i_ℓ} we create the permutation

$$\hat{\pi} = \pi_1 \cdots \pi_{i_\ell-1}(q+1)\pi_{i_\ell} \cdots \pi_q \in P(S;q+1).$$

Case 4.1: If $\pi \in P(S_{i_\ell};q)$ and $1 < \ell \leq s$, then π has a peak at position $i_{\ell-1}$ and by inserting $q+1$ between $\pi_{i_{\ell-1}-1}$ and $\pi_{i_{\ell-1}}$ we create the permutation

$$\hat{\pi} = \pi_1 \cdots \pi_{i_{\ell-1}-1}(q+1)\pi_{i_{\ell-1}} \cdots \pi_q \in P(S;q+1).$$

Case 4.2: If $\pi \in P(S_{i_1};q)$ where $S_{i_1} = \{i_1-1, i_2-1, \ldots, i_s-1\}$, then by inserting $q+1$ to the left of π_1 we create the permutation

$$\hat{\pi} = (q+1)\pi_1 \cdots \pi_q \in P(S;q+1).$$

Case 5: If $\pi \in P(\widehat{S}_{i_\ell};q)$ for any $1 \leq \ell \leq s$, then π has no peak at position i_ℓ. By inserting $q+1$ between $\pi_{i_\ell-1}$ and π_{i_ℓ} we create the permutation

$$\hat{\pi} = \pi_1 \cdots \pi_{i_\ell-1}(q+1)\pi_{i_\ell} \cdots \pi_q \in P(S;q+1).$$

The permutations $\hat{\pi}$ created via Cases 1 through 5 are distinct elements of $P(S;q+1)$. To see this note that if two permutations are the same, when you remove $q+1$ from each they will stay the same, hence they will have the same peak set. Thus the only potential collisions are between Cases 1 and 2, or between Cases 3 and 4. In both cases the permutations in question are distinct because $q+1$ appears in different positions.

In fact, we show that $P(S;q+1)$ is precisely the union of the permutations $\hat{\pi}$ appearing in Cases 1 through 5. If this is the case, the sets being disjoint gives us

$$|P(S;q+1)| = 2|P(S;q)| + 2\sum_{\ell=1}^{s} |P(S_{i_\ell};q)| + \sum_{\ell=1}^{s} |P(\widehat{S}_{i_\ell};q)|.$$

Note that any permutation $\hat{\pi}$ in $P(S;q+1)$ has the number $q+1$ in one of the following positions: $1, i_1, \ldots, i_s, q+1$. If $q+1$ is in position $q+1$, then removing it from the permutation $\hat{\pi}$ yields a permutation π in Case 1. If $q+1$ is in the first position, then removing it from the permutation $\hat{\pi}$ yields a permutation π in Case 4.2. If $q+1$ is in position i_ℓ for some $1 \leq \ell \leq s$, then removing it from the permutation $\hat{\pi}$ leads to three possibilities: a permutation with a peak at position i_ℓ (Cases 2 and 4.1), a permutation with a peak at position $i_\ell - 1$ (Case 3), or a permutation without a peak at positions $i_\ell - 1$ or i_ℓ (Case 5). Thus we have created all permutation in $P(S; q+1)$ via the constructions in Cases 1-5. $\square$

Note that the recurrence provided in Theorem 2.3 also holds whenever $S = \emptyset$ as the only contributing term is $2|P(\emptyset; q)|$. The following result plays a key role in the proof of Theorem 1.2.

Corollary 2.4. *Let $S = \{i_1, i_2, \ldots, i_s\} \subseteq [n]$ with $i_1 < i_2 < \ldots < i_s$ be a nonempty n-admissible set. Then the following equality of polynomials holds*

$$(\Delta p_S)(x) \;=\; \sum_{\ell=1}^{s} p_{S_{i_\ell}}(x) + \sum_{\ell=1}^{s} p_{\widehat{S}_{i_\ell}}(x). \tag{2.2}$$

Proof. Let $m = \max(S)$. It suffices to show that the two polynomials agree at infinitely many values, and to do so we show that for any $q \geq m$,

$$(\Delta p_S)(q) \;=\; \sum_{\ell=1}^{s} p_{S_{i_\ell}}(q) + \sum_{\ell=1}^{s} p_{\widehat{S}_{i_\ell}}(q). \tag{2.3}$$

Observe that for such q, substituting Equation (1.1) appropriately into Theorem 2.3 yields

$$\begin{aligned} 2^{q-|S|}p_S(q+1) - 2^{q-|S|}p_S(q) &= \sum_{\ell=1}^{s} 2^{q-|S_{i_\ell}|}p_{S_{i_\ell}}(q) + \sum_{\ell=1}^{s} 2^{q-|\widehat{S}_{i_\ell}|-1}p_{\widehat{S}_{i_\ell}}(q) \\ &= 2^{q-|S|}\sum_{\ell=1}^{s} p_{S_{i_\ell}}(q) + 2^{q-|S|}\sum_{\ell=1}^{s} p_{\widehat{S}_{i_\ell}}(q) \end{aligned} \tag{2.4}$$

where the last equality holds since $|S_{i_\ell}| = |S|$ and $|\widehat{S}_{i_\ell}| = |S| - 1$ for all $1 \leq \ell \leq s$. The result follows from multiplying Equation (2.4) by $1/2^{q-|S|}$. $\square$

We are now ready to prove the positivity conjecture for peak polynomials. We note that the proof presented in this manuscript does not consider certain edge cases, however we present a full detailed proof in the published version [10].

Proof of Theorem 1.2. We induct on $m = \max(S)$. The base case is when $S = \{2\}$. It is known that $p_{\{2\}}(x) = x - 2$ [2, Theorem 6]. Hence, we see $(\Delta p_{\{2\}})(x) = 1 > 0$, and $(\Delta^2 p_{\{2\}})(x) = 0$. Now suppose S is an arbitrary non-empty admissible set satisfying the conditions of the theorem, and further suppose the theorem holds for all peak polynomials $p_T(x)$ with admissible set T satisfying $\max(T) < m$. Let $S = \{i_1, i_2, \ldots, i_s\} \subseteq [n]$ with

$i_1 < i_2 < \ldots < i_s$, and for $1 \leq \ell \leq s$ construct the sets S_{i_ℓ} and $\widehat{S}_{i_\ell}$. From Corollary 2.4, we have

$$(\Delta p_S)(x) \;=\; \sum_{\ell=1}^{s} p_{S_{i_\ell}}(x) + \sum_{\ell=1}^{s} p_{\widehat{S}_{i_\ell}}(x).$$

For any $1 \leq j \leq m-1$,

$$(\Delta^j p_S)(x) \;=\; (\Delta^{j-1}(\Delta p_S))(x) \;=\; \sum_{\ell=1}^{s}(\Delta^{j-1} p_{S_{i_\ell}})(x) + \sum_{\ell=1}^{s}(\Delta^{j-1} p_{\widehat{S}_{i_\ell}})(x), \tag{2.5}$$

where $(\Delta^0 p_S)(x) = p_S(x)$. Let $k \geq m$. Since $\deg(p_T(x)) = \max(T) - 1$ for any nonempty admissible set $T \subseteq [n]$ and $\deg(p_T(x)) = 0$ when $T = \emptyset$, we have that for all $1 \leq \ell \leq s$

$$\deg(p_S(x)) > \deg(p_{\widehat{S}_{i_\ell}}(x)) \quad \text{and} \quad \deg(p_S(x)) > \deg(p_{S_{i_\ell}}(x)).$$

Since for all $1 \leq \ell \leq s$ we have $\max(S_{i_\ell}) = m-1$ then by induction it follows that for $k \geq m$

$$(\Delta^{j-1} p_{S_{i_\ell}})(k) > 0 \text{ for } 0 \leq j-1 \leq m-2.$$

If $|S| \geq 2$ then for all $1 \leq \ell \leq s$ we have $\max(\widehat{S}_{i_\ell}) < m$ and by induction it follows that for $k \geq m$

$$(\Delta^{j-1} p_{\widehat{S}_{i_\ell}})(k) > 0 \text{ for } 0 \leq j-1 \leq \max(\widehat{S}_{i_\ell}) - 1 \;\text{ and }\; (\Delta^{j-1} p_{\widehat{S}_{i_\ell}})(k) = 0 \text{ for } j-1 \geq \max(\widehat{S}_{i_\ell}).$$

Finally, when $S = \{i_s\}$ then $\widehat{S}_{i_s} = \emptyset$ and

$$(\Delta^{j-1} p_{\widehat{S}_{i_s}})(k) = (\Delta^{j-1} p_{\emptyset})(k) = 0 \text{ for } j-1 \geq 1 \quad \text{and} \quad (\Delta^0 p_{\widehat{S}_{i_s}})(k) = p_{\emptyset}(k) = 1.$$

From (2.5) we see that $(\Delta^j p_S)(k) > 0$. Finally, we claim that $(\Delta^m p_S)(x) = 0$. Since $\deg(p_S(x)) = m-1$ and the operator Δ decreases the degree by one, we see that $(\Delta^{m-1} p_S)(x) = c$ is a positive constant and $(\Delta^m p_S)(x) = 0$. □

With this conjecture proven in the affirmative we welcome a combinatorial description of the integers appearing in the peak polynomial binomial expansion.

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